

A REMARK ON THE CAPABILITY OF FINITE p -GROUPS

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ABSTRACT. In this paper we classify all capable finite p -groups with derived subgroup of order p and G/G' of rank $n - 1$.

1. MOTIVATION

Recall the famous question of P. Hall about a given group G “Can we decide that $G \cong H/Z(H)$ for a group H ?” That is an interesting question but unfortunately finding necessary and sufficient conditions for a group G to be isomorphic to $H/Z(H)$ is not easy. If it is possible such group is called capable following to [4]. It is known that which of finitely generated abelian groups are capable (see [2] for more information). Also, among the non-abelian groups, the capability of p -groups took special attention, although the structure of all non-abelian p -groups have not been characterized but the results of [1, 6] is determined the capability of two generator 2-groups.

In the preset paper we are interesting to classify the capable group where G' is of order p and G/G' of rank $n - 1$.

2. PRELIMINARIES

This section contains some definitions, theorems and lemmas which are used in main results. We assume that the notion of Schur multiplier is known, also we use the notion of epicenter and exterior center of a group without defining them. Epicenter of a group G which is denoted by $Z^*(G)$, was introduced by Beyl, Felgner, and Schmid in [3]. They showed a necessary and sufficient condition for a group to be capable is having trivial epicenter.

Theorem 2.1. (See [5, Theorem 2.5.10]) *Let G be a finite group and N be a normal subgroup of G . Then $N \subseteq Z^*(G)$ if and only if the natural map $M(G) \rightarrow M(G/N)$ is monomorphism.*

Theorem 2.2. (See [5, Theorem 2.2.10]) *Let A and B be finite groups then*

$$\mathcal{M}(A \times B) \cong \mathcal{M}(A) \oplus \mathcal{M}(B) \oplus \frac{A}{A'} \otimes \frac{B}{B'}.$$

Theorem 2.3. (See [5, Theorem 2.5.6 (i)]) *Let G be a finite group and N be a central subgroup of it, then the following sequence is exact*

$$\mathcal{M}(G) \rightarrow \mathcal{M}\left(\frac{G}{N}\right) \rightarrow N \cap G' \rightarrow 1.$$

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The following lemma is a conclusion of Theorem 2.3 and used in the proof of the main theorem.

Lemma 2.4. *Let G be a finite p -group and $N \subseteq Z(G) \cap G'$ be a subgroup of order p . If $|\mathcal{M}(G/N)| = p |\mathcal{M}(G)|$ then $N \subseteq Z^*(G)$.*

Proof. Using Theorems 2.1 and 2.3 it is enough to show that $\mathcal{M}(G) \rightarrow \mathcal{M}(G/N)$ has trivial kernel. Let α and β denote the homomorphisms $\mathcal{M}(G) \rightarrow \mathcal{M}(G/N)$ and $\mathcal{M}(G/N) \rightarrow N \cap G'$, respectively. Since $|N| = p$, we have $|\ker \beta| = |\mathcal{M}(G/N)|/p$ which is equal to $|\mathcal{M}(G)|$. Now Theorem 2.3 implies $\ker \alpha = 1$ as required. \square

The following lemma is a consequence of [5, Corollary 2.5.3].

Lemma 2.5. *Let G be a finite p -group then*

$$|\mathcal{M}(\frac{G}{\phi(G)})| \leq |\mathcal{M}(G)| |\phi(G) \cap G'|$$

3. MAIN RESULTS

Let G be a group of order p^n and G' is of order p and G/G' is elementary abelian. By [7, Lemma 2.1] we have $G = H \cdot Z(G)$ in which H is an extra special p -group and “ \cdot ” denotes the central product of groups. We know that $G' \subseteq Z(G)$, now depending on the structure of $Z(G)$ and the way that G' embeds in $Z(G)$ the structure of G may be simplified as the following theorem asserts.

Theorem 3.1. *Let $|G| = p^n$ and $Z(G)$ is not cyclic then*

- (1) *if for some K , $Z(G) = G' \oplus K$ then $G = H \times K$;*
- (2) *if G' is not a direct summand of $Z(G)$ then $G = (H \cdot \mathbb{Z}_{p^t}) \times K$ in which $Z(G) = \mathbb{Z}_{p^{t+1}} \oplus K$ and $G' \subseteq \mathbb{Z}_{p^{t+1}}$.*

Proof. (i) Since $G = H \cdot Z(G)$ and $H \cap Z(G) = G'$, so $G = H \times K$.

(ii) The proof is similar to the pervious part. \square

The main theorem of this paper is

Theorem 3.2. *Let $G = H \cdot Z(G)$ be the group as above, then G is capable if and only if H is capable and G' is a direct summand of $Z(G)$.*

The proof of the Main Theorem is partitioned into some cases as follows. From now on G is of order p^n and G/G' is elementary abelian of order p^{n-1} .

Theorem 3.3. *Let G be as above and $Z(G)$ is cyclic then G is not capable.*

Proof. Since G/G' is elementary abelian p -group we have $\phi(G) = G'$. Now using Lemma 2.5 and Main Theorems of [7, 8], we have

$$p^{\frac{1}{2}(n-1)(n-2)-1} \leq |\mathcal{M}(G)| \leq p^{\frac{1}{2}(n-1)(n-2)+1}.$$

Again using Main Theorems of [7, 8], we deduce that

$$|\mathcal{M}(G)| = p^{\frac{1}{2}(n-1)(n-2)-1},$$

so the following sequence is exact

$$1 \rightarrow \mathcal{M}(G) \rightarrow \mathcal{M}(\frac{G}{G'}) \rightarrow G' \rightarrow 1$$

which shows $G' \subseteq Z^*(G)$. \square

Theorem 3.4. *Let G be as above and G' be a direct summand of $Z(G)$ then G is capable if and only if H is capable.*

Proof. Theorem 3.1 shows in this case $G = H \times K$ where $Z(G) = G' \oplus K$. Remember that we have $Z^\wedge(G) \subseteq G' = H'$. Now depending on the capability of H we have the following cases

- (1) H is capable.
- (2) H is not capable.

In case (1), for $p \neq 2$, H is the extra special p -group of order p^3 and exponent p and that $|\mathcal{M}(H)| = p^2$. If we show that $H' \not\subseteq Z^*(G)$ we have done. To do this we use Theorems 2.1 and 2.3. The sequence

$$\mathcal{M}(G) \longrightarrow \mathcal{M}\left(\frac{G}{H'}\right) \longrightarrow H' \longrightarrow 1$$

is exact, but using Theorem 2.2 we have $|\mathcal{M}(G)| = p |\mathcal{M}(G/H')|$ so

$$1 \neq |\ker(\mathcal{M}(G) \longrightarrow \mathcal{M}(G/H'))|$$

and the result holds. For $p = 2$, G isomorphic to dihedral group of order 8, and a similar technique shows the result.

In case (2), H can be either an extra special p -group of order p^3 and exponent p^2 , or an extra special p -group of order p^{2m+1} with $m > 1$ which multipliers are trivial and of order p^{2m^2-m-1} , respectively. For H of order p^3 a similar argument to that of the first case shows that

$$\mathcal{M}(G) \longrightarrow \mathcal{M}\left(\frac{G}{H'}\right)$$

is injective and so $H' \subseteq Z^*(G)$. On the other hand if H is of order p^{2m+1} for $m > 1$, using Theorem 2.2 the following sequence is exact, therefore G is not capable.

$$1 \longrightarrow \mathcal{M}(G) \longrightarrow \mathcal{M}\left(\frac{G}{H'}\right) \longrightarrow H' \longrightarrow 1$$

□

Now the second case of Theorem 3.1 remains to discuss.

Theorem 3.5. *Let G be as above with $Z(G)$ not cyclic and G' is not a direct summand of $Z(G)$ then G is not capable.*

Proof. In this case we have $G = H \cdot \mathbb{Z}_{p^t} \times K$ for some $K \subseteq Z(G)$. Theorem 3.3 shows that $|\mathcal{M}(H \cdot \mathbb{Z}_{p^t})| = p^{1/2(t-1)(t-2)-1}$ and also $Z^*(H \cdot \mathbb{Z}_{p^t}) = (HC_{p^t})'$ is of order p . With the aid of Theorem 2.2, we compute the orders of the Schur multipliers of G and $G/(H \cdot \mathbb{Z}_{p^t})'$, we have

$$|\mathcal{M}\left(\frac{G}{(H \cdot \mathbb{Z}_{p^t})'}\right)| = p |\mathcal{M}(G)|.$$

Now the results follows by Lemma 2.4. □

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